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Procedia Computer Science 9 (2012) 2000 – 2003

Procedia Computer Science

International Conference on Computational Science, ICCS 2012

Semi-Coarsening in Space and Time for the Hierarchical Transformation Multigrid Method

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Abstract

We extend the hierarchical transformation multigrid (HT-MG) method with semi-coarsening in space and time in order to tackle anisotropic problems. Semi-coarsening allows us to smooth on anisotropic grids and, thus to employ more advanced multigrid cycles, e.g. the so-called Q-cycle. Our numerical examples show that we can tackle anisotropic problems with the HT-MG method and the Q-cycle. We do not only apply the multigrid method in space directions but also in time directions where the Q-cycle is beneficial again.

Keywords: anisotropic problems, multigrid, semi-coarsening, time-space discretization

1. Introduction

Multigrid methods [1, 2] have been shown to be among the fastest solution techniques for a wide class of partial differential equations. We consider the hierarchical transformation multigrid (HT-MG) method introduced in [3]. It uses a hierarchical representation of the solution function, i.e., the change of values at coarse grid points entails an appropriate change of the values at fine grid points. We represent each operator as a tensor product of one-dimensional operators which allows us to treat each coordinate direction separately. Thus, semi-coarsening becomes possible which means we can coarse and refine in each direction independently. This allows us to include anisotropic grids in our grid hierarchy. We also consider a space-time discretization of the heat equation, cf. [4] for a space-time HT-MG multigrid method with an emphasis on massively parallel machines and the usual multigrid cycles without semi-coarsening. Here we combine the HT-MG method with advanced multigrid cycles employing semi-coarsening and a space-time discretization which leads to a highly efficient method as we will show by numerical examples in Sec. 4.

2. Hierarchical Transformation Multigrid (HT-MG) Method

Let $\Omega \in \mathbb{R}^d$ be a domain with boundary $\partial\Omega = \Gamma$. We consider the linear boundary value problem

$$Lu = f \tag{1}$$

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doi:10.1016/j.procs.2012.04.221

with the operator $\mathrm{L} : U \to F$ and two appropriate function spaces U and F, see [1]. We assume that the operator L already incorporates the Dirichlet boundary conditions. We denote the discrete analog of $\mathrm{L}u = f$ with $\mathrm{L}_k u_k = f_k$ where $\mathrm{L}_k : U_k \to F_k$ is a linear operator and U_k, F_k are appropriate spaces of grid functions on the grid Ω_k with mesh width $h_k = 1/3^k$. Note that we use grids with $h_k = 1/3^k$ instead of the more common grids with $h_k = 1/2^k$, cf. [5].

Let u_k be an approximation of the solution of $\mathrm{L}_k u_k = f_k$ on the grid Ω_k. Let $\mathrm{P}_{k-1}^k : U_{k-1} \to U_k$, $\hat{\mathrm{R}}_k^{k-1} : U_k \to U_{k-1}$, $\hat{\mathrm{P}}_{k-1}^k : F_{k-1} \to F_k$ and $\mathrm{R}_k^{k-1} : F_k \to F_{k-1}$ be the prolongation and restriction operators. With the restriction $\hat{\mathrm{R}}_k^{k-1}$ we induce a coarse grid approximation $\hat{u}_{k-1} = \hat{\mathrm{R}}_k^{k-1} u_k$ on Ω_{k-1}. With the prolongation P_{k-1}^k we find the equation

$$u_k = \mathrm{P}_{k-1}^k \hat{u}_{k-1} + \overline{u}_k = \mathrm{P}_{k-1}^k \hat{\mathrm{R}}_k^{k-1} u_k + \overline{u}_k. \tag{2}$$

We derive the hierarchical surplus $\overline{u}_k$ of the fine grid approximation u_k as $\overline{u}_k = u_k - \mathrm{P}_{k-1}^k \hat{\mathrm{R}}_k^{k-1} u_k$. The transformation from u_k to $\mathrm{P}_{k-1}^k \hat{\mathrm{R}}_k^{k-1} u_k + \overline{u}_k$ is called the hierarchical transformation [3]. Fig. 1 shows the hierarchical transformation of an one-dimensional example. It illustrates, if coarse grid points are modified, the values at fine grid points change as well.

In the context of multigrid methods, we do not only need one discrete operator L_ℓ on the finest grid Ω_ℓ but discrete operators on the whole grid hierarchy Ω_k, $1 \leq k \leq \ell$. With the Galerkin approach $\mathrm{L}_{k-1} = \mathrm{R}_k^{k-1} \mathrm{L}_k \mathrm{P}_{k-1}^k$ and the hierarchical transformation (2) we obtain the coarse grid system

$$\mathrm{L}_{k-1} \hat{u}_{k-1} = \mathrm{R}_k^{k-1} f_k - \mathrm{R}_k^{k-1} \mathrm{L}_k \overline{u}_k.$$

The whole HT-MG algorithm can be summarized as follows (cf. [3]):

1. Smooth $\mathrm{L}_k u_k = f_k$ with ν relaxation sweeps
2. Hierarchical transformation $\overline{u}_k = u_k - \mathrm{P}_{k-1}^k \hat{\mathrm{R}}_k^{k-1} u_k$
3. Compute right hand side with $\hat{\mathrm{L}}_k^{k-1} = \mathrm{R}_k^{k-1} \mathrm{L}_k$ and $\hat{f}_{k-1} = \mathrm{R}_k^{k-1} f_k - \hat{\mathrm{L}}_k^{k-1} \overline{u}_k$
4. Initialize $\hat{u}_k = \hat{\mathrm{R}}_k^{k-1} u_k$ and call HT-MG with system $\mathrm{L}_{k-1} \hat{u}_k = \hat{f}_{k-1}$
5. Compute correction $u_k = \overline{u}_k + \mathrm{P}_{k-1}^k \hat{u}_{k-1}$
6. Smooth $\mathrm{L}_k u_k = f_k$ with ν relaxation sweeps

The order of recursive calls of the HT-MG algorithm depends on the employed multigrid cycle. A simple cycle s-Cycle(d, ν, S) performs one multigrid iteration with coarsening in direction d where ν relaxation sweeps S are used for pre- and postsmoothing. Note that we allow mesh size 1 in time in order to cope with the stationary heat equation (Poisson's equation). A V-cycle in space and time corresponds to the call "s-Cycle(t, 1, s-Cycle((x, y), ν, S))" which means that an ordinary V-cycle in space is used at every time step $0, \ldots, \ell_t$. Whereas the V-cycle coarsens in x and y direction simultaneously, the Q-cycle coarsens every direction separately, i.e. a Q-cycle corresponds to "s-Cycle(t, 1, s-Cycle(x, 1, s-Cycle(y, ν, S)))". This leads to grids with different mesh sizes in different directions, i.e. semi-coarsening. Fig. 1 visualizes how the V- and the Q-cycle traverse the grid hierarchy in the two space directions.

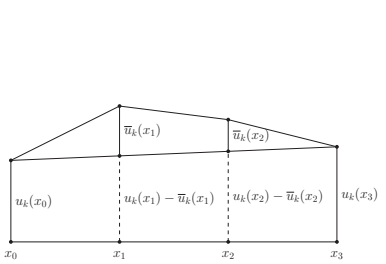
3. Discretization of the Heat Equation in Space and Time

Let us now come to a suitable discretization of the heat equation $\partial_t u - \Delta u = f$ for the HT-MG method. The discretization has to be carried out in both space and time. We look upon time just as any other coordinate direction. Let $\Omega \subseteq \mathbb{R}^2$ be a domain and $(0, t_{\text{end}}) \subseteq \mathbb{R}$ the time interval of length t_{end}. The heat equation is

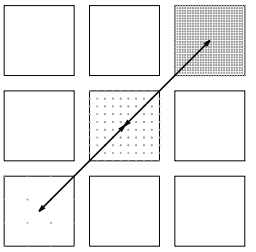
$$\begin{aligned} a \cdot \partial_t u(x,y,t) - \left(b_x \cdot \partial_x^2 u(x,y,t) + \partial_y^2 u(x,y,t)\right) &= f(x,y,t), & (x,y,t) &\in \Omega \times (0, t_{\text{end}}), \\ u(x,y,t) &= 0, & (x,y,t) &\in \partial\Omega \times (0, t_{\text{end}}), \\ u(x,y,0) &= u_0(x,y), & (x,y) &\in \Omega, \end{aligned}$$

where $u : \overline{\Omega} \times [0, t_{\text{end}}] \to \mathbb{R}^2$, $u_0 : \Omega \to \mathbb{R}^2$ and an appropriate right hand side $f \in L^2(\Omega \times (0, t_{\text{end}}))$. Note that we obtain Poisson's equation with $a = 0$ and an anisotropic problem with $b_x \gg 1$. In contrast to the discretization given in [3], our discretization considers every operator separately and not the whole equation at once. Each operator is split into a tensor product of one-dimensional operators

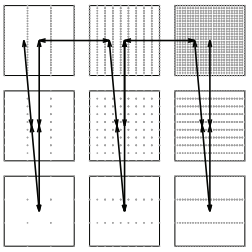
$$\partial_t u(x,y,t) - \left(\partial_x^2 u(x,y,t) + \partial_y^2 u(x,y,t)\right) = \left(\mathrm{I}_x \times \mathrm{I}_y \times \mathrm{D}_t\right) u(x,y,t) - \left(\mathrm{D}_x^2 \times \mathrm{I}_y \times \mathrm{I}_t + \mathrm{I}_x \times \mathrm{D}_y^2 \times \mathrm{I}_t\right) u(x,y,t),$$



(a) Hier. Transformation



(b) V-cycle



(c) Q-cycle

Figure 1: Hierarchical transformation (a) of u_k with the coarse grid points x_0 and x_3 and the fine grid points x_1 and x_2. Note that the hierarchical surplus at a coarse grid point is always 0 because we employ linear prolongation and trivial restriction. The HT-MG method can employ either the V-cycle (b) which refines in both space directions simultaneously or the Q-cycle (c) which refines each direction separately.

where D denotes the derivation and I the identity operator. We now need a discretization of the one-dimensional operators D and I, and in order to allow to coarse and refine in each direction separately, we also need their coarse counter parts. Such one-dimensional operators are constructed in [6] and lead to the well-known stencil $1/6 \cdot [1\ 4\ 1]$ for I and $1/h^2 \cdot [1\ -2\ 1]$ for D. With the help of an eigenvalue analysis, it can be shown, that the stencils only change by a factor of 3 during a coarsening or refinement step, see [6] for details. This means, we can keep the stencils during the whole multigrid cycle and only have to take care that the result is multiplied with the correct power of 3.

4. Numerical Results

We consider the heat equation and Poisson's equation with the discretization of Sec. 3. We always use the lexicographical Gauss-Seidel relaxation [1] to smooth the error, as well as the trivial injection as restriction, and linear interpolation as prolongation in space directions and constant interpolation in time direction. Note that we can have a different prolongation in time than in space directions [6]. If not otherwise specified, we have the right hand side $f = 1$.

First, we consider stationary ($a = 0$) anisotropic problems ($b_x \gg 1$). Fig. 2 (left) shows the convergence of the HT-MG method with the V-cycle with $b_x \in \{1, 10, 50, 100\}$ and $\nu = 2$ pre- and postsmoothing steps on the grid $243 \times 243 \times 1$ with mesh size $h = 1/243$. The V-cycle works well for the isotropic problem ($b_x = 1$) but already for the anisotropic problem with only $b_x = 10$, we observe a very slow convergence. This becomes even worse with $b_x = 50$ and $b_x = 100$. It is well known that we can recover multigrid performance with semi-coarsening [1]. Thus, we employ the Q-cycle with semi-coarsening, see Fig. 2 (right) and we obtain a rapid convergence of the HT-MG method not only with $b_x = 1$ but also with $b_x \in \{10, 50, 100\}$.

We also want to compare the V- and the Q-cycle with respect to the number of relaxations needed to obtain a certain accuracy. It can be easily shown that during one Q-cycle three to four times as many equations are relaxed as during one V-cycle. The convergence behavior of the HT-MG method with the V-cycle and $b_x = 50$ is shown in Fig. 2 (left). About 10 V-cycles are needed to reduce the residual by a factor of 10^2. However, we can achieve the same reduction with only one Q-cycle. Thus, the Q-cycle is also distinctly more efficient with respect to the number of relaxations.

A similar behavior can be observed for the heat equation ($a = 1, f = 1$) if we employ the HT-MG method in space and time on the grid $243 \times 243 \times 9$. We consider $b_x \in \{1, 10, 50, 100\}$ but now with $a = 1$. We employ the V- and Q-cycle not only in space directions but in space and time directions. Fig. 3 compares the performance of the V- and the Q-cycle in space and time. Whereas the V-cycle has distinct problems, the Q-cycle can also cope with anisotropic problems with $b_x \gg 1$ again.

5. Conclusion

We added semi-coarsening (Q-cycle) to the hierarchical transformation multigrid (HT-MG) method introduced in [3]. With semi-coarsening it is possible to employ more advanced multigrid cycles, e.g. the Q-cycle, which do

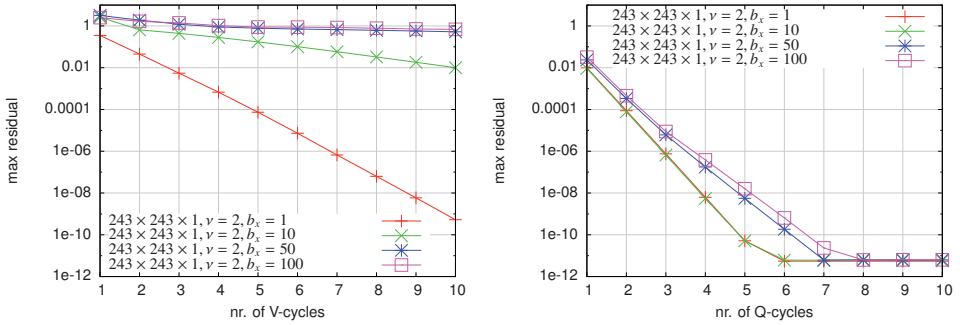


Figure 2: Stationary ($a = 0$) problem with $b_x \in \{1, 10, 50, 100\}$: the V-cycle (left) cannot cope with anisotropic problems ($b_x \gg 1$). However, we can tackle such problems with the HT-MG method if we employ the Q-cycle and semi-coarsening (right).

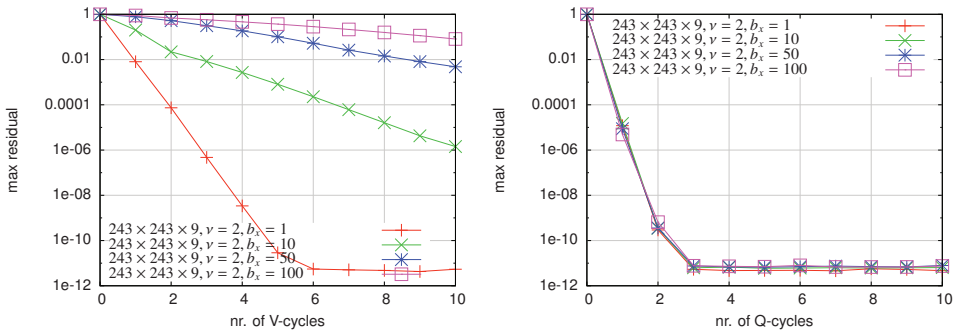


Figure 3: Heat equation ($a = 1$) with $b_x \in \{1, 10, 50, 100\}$: Again, only with the Q-cycle (right) and semi-coarsening, the HT-MG method shows a rapid convergence for problems with $b_x \gg 1$.

not only use a grid hierarchy with isotropic but also with anisotropic grids. Such a grid hierarchy is needed for e.g. anisotropic problems which play an important role in practice [1]. In order to allow semi-coarsening, it was necessary to split the operators of the governing equation into tensor products of one-dimensional operators. The numerical results have shown that anisotropic problems can be efficiently treated with the Q-cycle. This is not only true for multigrid in space directions, but also for multigrid in space and time.

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